

On quantum subsystem measurement

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Abstract. It is assumed that an arbitrary composite bipartite pure state in which the two subsystems are entangled is given, and it is investigated how the entanglement transmits the influence of measurement on only one of the subsystems to the state of the opposite subsystem. It is shown that any exact subsystem measurement has the same influence as ideal measurement on the opposite subsystem. In particular, the distant effect of subsystem measurement of a twin observable, i. e., so-called 'distant measurement', is always ideal measurement on the distant subsystem no matter how intricate the direct exact measurement on the opposite subsystem is.

Keywords Entanglement in measurement. Measurement effects due to entanglement. Unitary measurement. Basic dynamics.

1 Introduction

The present article investigates some implications of defining the measuring process by a **unitary operator** that incorporates the interaction between object and measuring instrument. One deals with so-called **nonselective measurement**, i. e., measurement short of collapse (if done on an ensemble, this contains all the results). So-called **selective measurement** is measurement with collapse, when one result is considered (the subensemble of this result is selected). The mechanism of collapse is known to lie outside unitary dynamics [1]. It will not be considered in this study. Most interpretations of collapse are in agreement with the quantum-mechanical formalism, which implies the unitary measurement dynamics presented.

In the literature by measurement one usually means selective measurement. In this article we mean by measurement nonselective measurement unless otherwise stated.

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The terms 'pure state' and 'state vector' (vector of norm one) will be used interchangeably; and so will 'state' and 'density operator', 'observable' and 'Hermitian operator' (with a purely discrete spectrum), event' and 'projector' throughout the paper. The subsystem over which a partial trace is taken will be denoted by index (or indices). Total traces go without indices.

Let subsystem A be the object of measurement, and let

$$O_A = \sum_k o_k E_A^k, \quad k \neq k' \Rightarrow o_k \neq o_{k'} \quad (1)$$

be **the measured observable** (Hermitian operator with a purely discrete, finite or infinite, spectrum) **in its unique spectral form**. By 'uniqueness' is meant the non-repetition of the eigenvalues $\{o_k : \forall k\}$ in (1)). Henceforth, we always mean by 'spectral form' the unique one unless otherwise stated.

Naturally, also the completeness relation $\sum_k E_A^k = I_A$, I_A being the identity operator for subsystem A, is valid. Let, further, subsystem B be the measuring instrument equipped with a pointer observable

$$P_B = \sum_k p_k F_B^k, \quad (2)$$

also in its spectral form. The completeness relation $\sum_k F_B^k = I_B$ is valid too.

The measuring apparatus 'takes cognizance' of the results, eigenvalues o_k or, equivalently, of the corresponding eigen-events E_A^k , in terms of its 'pointer positions', which are either the eigenvalues p_k of the pointer observable or, equivalently, the eigen-events F_B^k . (This is stated more precisely below when measurement is defined.)

Finally, let U_{AB} be the unitary operator incorporating the **measurement interaction** and mapping any initial composite-system state vector $|\phi\rangle_A \otimes |\phi\rangle_B^i$ into the **final state** (at the end of measurement interaction):

$$|\Phi\rangle_{AB}^f \equiv U_{AB}(|\phi\rangle_A |\phi\rangle_B^i). \quad (3)$$

By $|\phi\rangle_A$ is denoted an arbitrary state vector of the measured system A, and $|\phi\rangle_B^i$ is **the initial or ready-to-measure state vector** of the instrument.

We use the convention that kets and bras denote state vectors.

In this investigation the basic aim is to focus attention on **bipartite composite systems** in some pure state $|\Phi\rangle_{A_1 A_2}$ where $A \equiv A_1 + A_2$ is the object of measurement. We are particularly interested in **subsystem measurements** on subsystem A_2 , which we call the **nearby** subsystem, and on its influence on the opposite, dynamically unaffected subsystem A_1 , called **distant** or **remote**. (The terms are dynamical, not spatial.) The influence is transmitted by the **entanglement** in the composite state.

2 Definition and Basic Dynamical Property of Measurement

Exact measurement is defined by requiring the validity of the so-called **calibration condition** [2]. It reads: If the initial state of the object has a definite value of the measured observable, then the final composite-system state has the **corresponding** definite value of the pointer observable. 'Corresponding' we write as 'having the same index value' (cf (1) and (2)).

Since approximate measurements are not studied in this article, henceforth we drop the term 'exact'.

All quantum-mechanical relations have a statistical meaning and are tested on ensembles of equally prepared systems. The precise **statistical form** of the **calibration condition** is expressed in terms of the usual **probability formulae**:

$$\forall k : \langle \phi |_A E_A^k | \phi \rangle_A = 1 \Rightarrow \langle \Phi |_{AB}^f F_B^k | \Phi \rangle_{AB}^f = 1, \quad (4)$$

where \Rightarrow denotes logical implication, and the final state $|\Phi\rangle_{AB}^f$ is given by (3).

To derive an equivalent, more practical, form of (4), we need a useful general and known, but perhaps not well known, auxiliary **claim** (proved in Appendix A for the reader's convenience).

An event E is certain, i. e., has probability one, in a pure state $|\psi\rangle$ if and only if the former, acting on the latter, does not change it:

$$\langle \psi | E | \psi \rangle = 1 \Leftrightarrow E | \psi \rangle = | \psi \rangle. \quad (5)$$

(The symbol " \Leftrightarrow " denotes logical implication in both directions.)

Equivalence (5) makes it obvious that the calibration condition can be equivalently expressed in the more practical form:

$$\forall k : \quad |\phi\rangle_A = E_A^k |\phi\rangle_A \quad \Rightarrow \quad |\Phi\rangle_{AB}^f = F_B^k |\Phi\rangle_{AB}^f \quad (6)$$

(cf (1)-(3)).

Now we state and prove the **basic dynamical property** of measurement. Actually, it is a necessary and sufficient condition for the calibration condition, or otherwise put, it is another definition of measurement. (We call it "dynamical" because it involves the unitary evolution operator U_{AB} explicitly.) The **claim** goes as follows.

One has measurement **if and only if**

$$\forall |\phi\rangle_A, \forall k : \quad (F_B^k U_{AB}) (|\phi\rangle_A |\phi\rangle_B^i) = (U_{AB} E_A^k) (|\phi\rangle_A |\phi\rangle_B^i) \quad (7)$$

is valid.

One *proves necessity* as follows. The completeness relation $\sum_{k'} E_A^{k'} = I_A$, repeated use of the calibration condition (6), and orthogonality and idempotency of the F_B^k projectors enable one to write for each k value (we shall put \times after a number whenever a term in an expansion begins by that number):

$$\begin{aligned} & F_B^k U_{AB} |\phi\rangle_A |\phi\rangle_B^i = \\ & \sum_{k'} ||E_A^{k'} |\phi\rangle_A|| \times F_B^k U_{AB} (E_A^{k'} |\phi\rangle_A / ||E_A^{k'} |\phi\rangle_A||) |\phi\rangle_B^i = \\ & \sum_{k'} ||E_A^{k'} |\phi\rangle_A|| \times F_B^k \mathbf{F}_B^{k'} U_{AB} (E_A^{k'} |\phi\rangle_A / ||E_A^{k'} |\phi\rangle_A||) |\phi\rangle_B^i = \\ & ||E_A^k |\phi\rangle_A|| \times F_B^k U_{AB} (E_A^k |\phi\rangle_A / ||E_A^k |\phi\rangle_A||) |\phi\rangle_B^i. \end{aligned}$$

Finally, on account of (6) the auxiliary claim (5) allows one to omit F_B^k , so that, after cancelation, one obtains:

$$lhs = U_{AB} E_A^k |\phi\rangle_A |\phi\rangle_B^i.$$

To *prove sufficiency*, let

$$(U_{AB} E_A^k) (|\phi\rangle_A |\phi\rangle_B^i) = (F_B^k U_{AB}) (|\phi\rangle_A |\phi\rangle_B^i)$$

be valid for all k values, and let $|\phi\rangle_A = E_A^{k'} |\phi\rangle_A$ be satisfied for a fixed value $k \equiv k'$. Then, one has in particular

$$(U_{AB} E_A^{k'}) (|\phi\rangle_A |\phi\rangle_B^i) = (F_B^{k'} U_{AB}) (|\phi\rangle_A |\phi\rangle_B^i).$$

One can here omit $E_A^{k'}$ due to the assumed definite value using (5), and thus the calibration condition (6) is obtained. *This ends the proof.*

3 Subsystem Measurement in Composite State

In this section we assume that an arbitrary composite bipartite system $A \equiv A_1 + A_2$ in an arbitrary pure state $|\phi\rangle_{A_1, A_2}$ and an arbitrary subsystem observable $O_{A_2} = \sum_k o_k E_{A_2}^k$ for the nearby subsystem are given. We investigate the consequences of the basic dynamical characterization of measurement (7) for this case to find out how entanglement transmits the subsystem measurement dynamics on the nearby subsystem A_2 onto the state of the remote opposite subsystem A_1 .

To begin with, it is known that any unitary change to subsystem A_2 , with or without an ancilla A_3 , does not have any influence on the state of subsystem A_1 .

More precisely, the **claim** is that, if there is no interaction between subsystems A_1 and $A_2 + A_3$, i. e., if the composite unitary evolution operator can be factorized $U_{A_1, A_2, A_3} = U_{A_1} \otimes U_{A_2, A_3}$, then the **final remote subsystem state** reads

$$\rho_{A_1}^f \equiv \text{tr}_{A_2, A_3} (U_{A_1, A_2, A_3} |\phi\rangle_{A_1, A_2, A_3} \langle\phi|_{A_1, A_2, A_3} U_{A_1, A_2, A_3}^\dagger) = U_{A_1} \rho_{A_1}^i U_{A_1}^\dagger, \quad (8)$$

where $\rho_{A_1}^i \equiv \text{tr}_{A_2, A_3} (|\phi\rangle_{A_1, A_2, A_3} \langle\phi|_{A_1, A_2, A_3})$ is the initial state of subsystem A_1 in the composite-system state $|\phi\rangle_{A_1, A_2, A_3} = |\phi\rangle_{A_1, A_2} |\phi\rangle_{A_3}$.

Note that what makes the ancilla A_3 an auxiliary system is the fact that it is initially uncorrelated with the system $A_1 + A_2$ that is considered. Further, one should note that if there is no interaction with the ancilla, then the ancilla evolves independently, and it can be disregarded.

Though claim (8) is known, for the reason of completeness, we sketch the proof. But for this we need a general auxiliary claim, which will be referred to as the '**under-the-partial-trace commutativity**' (it will be used again below). It reads:

$$O_A \equiv \text{tr}_B (\mathbf{Y}_B X_{AB}) = \text{tr}_B (X_{AB} \mathbf{Y}_B), \quad (9)$$

where Y_B and X_{AB} are arbitrary subsystem and composite-system operators respectively. This general **claim** is proved in Appendix B.

Proof for claim (8) follows immediately from the definition in (8) when one takes into account the facts (i) that opposite-subsystem operators can be taken out of the partial trace preserving the order of the operators (U_{A_1} and $U_{A_1}^\dagger$ in this case), (ii) that one has the under-the-partial-trace commutativity (9), which concerns U_{A_2, A_3} with the rest, and finally, (iii) that a unitary operator (U_{A_2, A_3} in this case) multiplied by its inverse gives the identity operator. *This ends the proof.*

Since a measurement instrument B qualifies for an ancilla (cf (3)), though its role is far from auxiliary, it is clear from claim (8) that **nonselective measurement** of any nearby subsystem observable O_{A_2} in any pure state of a composite system $A_1 + A_2$ **cannot influence the state** of the distant subsystem A_1 .

Next, we are interested in **selective subsystem measurement**. The **general claim**, a consequence of the basic dynamical relation (7), goes as follows.

Selective measurement does, in general, influence the state of the remote subsystem A_1 . More precisely, if a nearby-subsystem observable $O_{A_2} = \sum_k o_k E_{A_2}^k$ is measured selectively with the result o_k in a bipartite pure state $|\phi\rangle_{A_1, A_2}$ in which one has positive probability $\langle\phi|_{A_1, A_2} E_{A_2}^k |\phi\rangle_{A_1, A_2} > 0$, then the **final selective distant-subsystem state**

$$\rho_{A_1}^{f,k} \equiv \text{tr}_{A_2, B} \left[\left(F_B^k |\Phi\rangle_{A_1, A_2, B}^f / \|F_B^k |\Phi\rangle_{A_1, A_2, B}^f\| \right) \left(\langle\Phi|_{A_1, A_2, B}^f F_B^k / \|F_B^k |\Phi\rangle_{A_1, A_2, B}^f\| \right) \right] \quad (10)$$

has the form:

$$\rho_{A_1}^{f,k} = U_{A_1} \left(\rho_{A_1}(E_{A_2}^k) \right) U_{A_1}^\dagger, \quad (11a)$$

where by

$$\rho_{A_1}(G_{A_2}) \equiv \text{tr}_{A_2} \left((|\phi\rangle_{A_1, A_2} \langle\phi|_{A_1, A_2}) G_{A_2} \right) / \text{tr} \left((|\phi\rangle_{A_1, A_2} \langle\phi|_{A_1, A_2}) G_{A_2} \right) \quad (11b)$$

(G_{A_2} being any projector in the state space \mathcal{H}_{A_2}) is denoted the **conditional state** of the remote subsystem A_1 under the condition of the

occurrence of the event G_{A_2} in the composite-system state $|\phi\rangle_{A_1,A_2}$, and U_{A_1} is the unitary evolution operator of the remote subsystem.

To *prove* (11a), we evaluate $\rho_{A_1}^{f,k}$ from its definition (10). By this we utilize the following equalities, which are a consequence of (7) and (3), of the fact that a unitary operator does not change the norm, and finally of the fact that the norm of a tensor product is the product of the norms.

$$\begin{aligned} \|F_B^k |\Phi\rangle_{A_1,A_2,B}^f\| &= \|E_{A_2} |\phi\rangle_{A_1,A_2}\| = \\ & \left(\langle \phi |_{A_1,A_2} E_{A_2}^k |\phi\rangle_{A_1,A_2} \right)^{1/2} = \left[\text{tr} \left((|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2}) E_{A_2}^k \right) \right]^{1/2}. \end{aligned} \quad (12)$$

Besides (12), we take again resort to (7), take into account the partial-trace property that opposite-subsystem operators can be taken out of the partial trace (preserving the order of the operators as factors), as well as the 'under-the-partial-trace commutativity' (9) twice:

$$\begin{aligned} \rho_{A_1}^{f,k} &= \left(\langle \phi |_{A_1,A_2} E_{A_2}^k |\phi\rangle_{A_1,A_2} \right)^{-1} \times \\ & \text{tr}_{A_2,B} \left[\left(U_{A_1} U_{A_2,B} E_{A_2}^k (|\phi\rangle_{A_1,A_2} |\phi\rangle_B^i) \right) \left((|\phi\rangle_{A_1,A_2} \langle \phi|_B^i) E_{A_2}^k U_{A_1}^\dagger U_{A_2,B}^\dagger \right) \right] = \\ & U_{A_1} \left\{ \text{tr}_{A_2,B} \left[\left(E_{A_2}^k (|\phi\rangle_{A_1,A_2} |\phi\rangle_B^i \langle \phi|_{A_1,A_2} \langle \phi|_B^i) E_{A_2}^k \right) \left(U_{A_2,B}^\dagger U_{A_2,B} \right) \right] \right\} U_{A_1}^\dagger / \\ & \text{tr} \left((|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2}) E_{A_2}^k \right) = \\ & U_{A_1} \left\{ \left[\text{tr}_{A_2} \left(E_{A_2}^k (|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2} E_{A_2}^k) \right) \right] \left[\text{tr}_B \left(|\phi\rangle_B^i \langle \phi|_B^i \right) \right] \right\} U_{A_1}^\dagger / \\ & \text{tr} \left((|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2}) E_{A_2}^k \right) = \\ & U_{A_1} \left[\text{tr}_{A_2} \left((|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2}) E_{A_2}^k \right) \right] U_{A_1}^\dagger / \text{tr} \left((|\phi\rangle_{A_1,A_2} \langle \phi|_{A_1,A_2}) E_{A_2}^k \right) = \\ & U_{A_1} \rho_{A_1} (E_{A_2}^k) U_{A_1}^\dagger. \end{aligned}$$

This ends the proof.

It is important to note that claim (11a) implies that it is irrelevant what kind of measurement is performed on the nearby subsystem, the effect on the distant subsystem is **one and the same**, and the influence of the measurement goes only in terms of the **eigen-projectors** of the measured observable. Another way to express this fact is to say that any measurement on the nearby subsystem acts on the distant subsystem equally as the simplest, i. e., **ideal measurement**.

Consistency of no change in nonselective measurement on the one hand, and of the evaluated change in selective measurement on the other, i. e., of (8) and (11a), is seen in the following decomposition.

$$\rho_{A_1}^i = \sum_k \left(\langle \phi |_{A_1, A_2} E_{A_2}^k | \phi \rangle_{A_1, A_2} \right) \rho_{A_1}(E_{A_2}^k). \quad (13)$$

To *prove* decomposition (13), we make use of the completeness relation $\sum_k E_{A_2}^k = I_{A_2}$ and of (12):

$$\begin{aligned} \rho_{A_1}^i = \sum_k \left(\langle \phi |_{A_1, A_2} E_{A_2}^k | \phi \rangle_{A_1, A_2} \right) \times \left\{ \text{tr}_{A_2} \left(| \phi \rangle_{A_1, A_2} \langle \phi |_{A_1, A_2} E_{A_2}^k \right) / \right. \\ \left. \left[\text{tr} \left(| \phi \rangle_{A_1, A_2} \langle \phi |_{A_1, A_2} E_{A_2}^k \right) \right] \right\}. \end{aligned}$$

In view of (11b), *this ends the proof*.

One should note that any orthogonal projector decomposition of the identity operator I_{A_2} induces likewise a decomposition of $\rho_{A_1}^i$ (displays the density operator as an improper mixture [3]). For the measurement of $O_{A_2} = \sum_k o_k E_{A_2}^k$ one of this mixtures, particularly (13), is relevant.

Relation (11a) tells us that all that selective nearby-subsystem measurement with the result o_k accomplishes on the remote subsystem is that it picks the state $\rho_{A_1}(E_{A_2}^k)$ in the corresponding mixture (13). In view of (8), the state $\rho_{A_1}(E_{A_2}^k)$ then evolves according to the dynamics of the remote subsystem with no regard to the chosen measurement on the nearby system.

This insight might be useful for any theory of collapse, i. e., of selective measurement.

4 Subsystem Measurement of Twin Observable; Distant Measurement

Now we assume that, for a given bipartite pure state $|\phi\rangle_{A_1, A_2}$, a pair of (opposite subsystem) **twin observables** O_{A_1} and O_{A_2} are given. By definition, they can be written as

$$O_{A_q} = \sum_k o_k^{(q)} E_{A_q}^k + O'_{A_q}, \quad q = 1, 2, \quad (14a, b)$$

where the the sums are written as unique spectral forms, and also

$$\forall k : \quad E_{A_1}^k |\phi\rangle_{A_1, A_2} = E_{A_2}^k |\phi\rangle_{A_1, A_2}; \quad (14c)$$

$$O'_{A_q} |\phi\rangle_{A_1, A_2} = 0, \quad q = 1, 2 \quad (14d)$$

are valid (cf [4]).

The following **claim** holds true. If only O_{A_2} of the above pair of **twin observables** is measured selectively on the nearby subsystem with the result $o_k^{(2)}$, then the final state of the remote subsystem is

$$\rho_{A_1}^{f,k} = U_{A_1} \left\{ E_{A_1}^k \rho_{A_1}^i E_{A_1}^k / \left[\text{tr}(\rho_{A_1}^i E_{A_1}^k) \right] \right\} U_{A_1}^\dagger, \quad (15)$$

and this is valid for every value of k .

To *prove* claim (15), we make use of (11b), of idempotency, of under-the-partial-trace commutativity, of the twin-observables definition (14c), and finally of the possibility to take out opposite-subsystem operators from the partial trace:

$$\begin{aligned} \rho_{A_1}(E_{A_2}^k) &\equiv \text{tr}_{A_2} \left((|\phi\rangle_{A_1, A_2} \langle \phi|_{A_1, A_2}) E_{A_2}^k \right) / \left[\text{tr} \left((|\phi\rangle_{A_1, A_2} \langle \phi|_{A_1, A_2}) E_{A_2}^k \right) \right] = \\ &\text{tr}_{A_2} \left((E_{A_2}^k |\phi\rangle_{A_1, A_2}) (\langle \phi|_{A_1, A_2} E_{A_2}^k) \right) / \left[\text{tr} \left((|\phi\rangle_{A_1, A_2} \langle \phi|_{A_1, A_2}) E_{A_2}^k \right) \right] = \\ &\text{tr}_{A_2} \left((\mathbf{E}_{\mathbf{A}_1}^k |\phi\rangle_{A_1, A_2}) (\langle \phi|_{A_1, A_2} \mathbf{E}_{\mathbf{A}_1}^k) \right) / \left\{ \text{tr}_{A_1} \left[\text{tr}_{A_2} \left(|\phi\rangle_{A_1, A_2} \langle \phi|_{A_1, A_2} \right) \right] \mathbf{E}_{\mathbf{A}_1}^k \right\} = \\ &E_{A_1}^k \rho_{A_1}^i E_{A_1}^k / \left[\text{tr}(\rho_{A_1}^i E_{A_1}^k) \right]. \end{aligned}$$

In view of (11a), *this ends the proof*.

The change of state

$$\rho_{A_1}^i \rightarrow E_{A_1}^k \rho_{A_1}^i E_{A_1}^k / \text{tr}(\rho_{A_1}^i E_{A_1}^k) \quad (16a)$$

is the well-known Lüders selective change-of-state formula (cf [5], [6], [7]), which characterizes **ideal selective measurement**.

One should note that $\text{tr}(\rho_{A_1}^i E_{A_1}^k) = \langle \phi |_{A_1, A_2} \mathbf{E}_{\mathbf{A}_2}^k | \phi \rangle_{A_1, A_2}$ (cf (12)) is the probability of the result $o_k^{(2)}$. Hence, the nonselective version of the same subsystem measurement on the nearby subsystem A_2 gives rise to

$$\sum_k \text{tr}(\rho_{A_1}^i E_{A_1}^k) \left[E_{A_1}^k \rho_{A_1}^i E_{A_1}^k / \text{tr}(\rho_{A_1}^i E_{A_1}^k) \right] = \sum_k E_{A_1}^k \rho_{A_1}^i E_{A_1}^k. \quad (16b)$$

This is not distinct from $\rho_{A_1}^i$ because the completeness relation $\sum_k E_{A_1}^k = I_{A_1}$ implies $\rho_{A_1}^i = \sum_{k, k'} E_{A_1}^k \rho_{A_1}^i E_{A_1}^{k'}$, and, for $k \neq k'$, one has on account of the twin relation (14c), under-the-partial-trace commutativity, and orthogonality of the eigen-projectors:

$$E_{A_1}^k \rho_{A_1}^i E_{A_1}^{k'} \equiv \text{tr}_{A_2} \left(E_{A_1}^k | \phi \rangle_{A_1, A_2} \langle \phi |_{A_1, A_2} E_{A_1}^{k'} \right) = \text{tr}_{A_2} \left(\mathbf{E}_{\mathbf{A}_2}^k | \phi \rangle_{A_1, A_2} \langle \phi |_{A_1, A_2} \mathbf{E}_{\mathbf{A}_2}^{k'} \right) = \text{tr}_{A_2} \left(| \phi \rangle_{A_1, A_2} \langle \phi |_{A_1, A_2} (E_{A_2}^{k'} E_{A_2}^k) \right) = 0.$$

Naturally, the fact that nonselective subsystem measurement of a twin observable on the nearby subsystem causes no change in the state of the distant subsystem is a special case of the general statement that every nearby subsystem measurement behaves in this way (that is proved in claim (8)).

Result (15) can be read in the following manner: An instantaneous ideal measurement of $\mathbf{O}_{\mathbf{A}_1}$ appears to be performed on the initial distant-subsystem state $\rho_{A_1}^i$, and then the state evolves in its unitary way till the end of the measurement of O_{A_2} on the nearby subsystem. The defining relations (11c) immediately implied this statement for ideal measurement on subsystem A_2 . Now, on account of the claim (11a), which covers **all** measurements on the nearby subsystem, we have the general validity of the statement.

The notion of distant measurement, introduced in [8], covered only the case when ideal subsystem measurement was performed on the nearby subsystem and it gave rise to ideal measurement on the remote subsystem (without interaction, only due to the entanglement). Since one rarely succeeds to perform ideal measurement in direct interaction, the distant-measurement

concept was thus on feet of clay. Now the notion of **distant measurement** is on firm ground: Any measurement of a twin observable \mathbf{O}_{A_2} (cf (14a-d)) on the nearby subsystem brings about **distant**, i. e., interaction free, **ideal measurement** of its twin observable \mathbf{O}_{A_1} on the opposite, remote subsystem.

Appendix A. Relation of certainty in a pure state

We *prove* now the general claim that the following equivalence is valid for a pure state $|\psi\rangle$ and an event E :

$$\langle\psi| E |\psi\rangle = 1 \quad \Leftrightarrow \quad |\psi\rangle = E |\psi\rangle.$$

One can write

$$\langle\psi| E |\psi\rangle = 1 \quad \Rightarrow \quad \langle\psi| E^c |\psi\rangle = 0,$$

where $E^c \equiv I - E$ is the ortho-complementary projector and I is the identity operator. Further, one has $\|E^c |\psi\rangle\| = 0$, $E^c |\psi\rangle = 0$, and $E |\psi\rangle = |\psi\rangle$ as claimed.

Appendix B. Under-the-partial-trace commutativity

We prove now the general relation

$$\text{tr}_B(Y_B X_{AB}) = \text{tr}_B(X_{AB} Y_B)$$

(cf (9)) by straightforward evaluation of both sides in an arbitrary pair of complete orthonormal bases $\{|k\rangle_A : \forall k\}$, $\{|n\rangle_B : \forall n\}$.

$$\begin{aligned} \langle k |_A lhs | k' \rangle_A &= \sum_n \langle k |_A \langle n |_B (Y_B X_{AB}) | k' \rangle_A | n \rangle_B = \\ \sum_n \sum_{k''} \sum_{n'} &\langle k |_A \langle n |_B (I_A \otimes Y_B) | k'' \rangle_A | n' \rangle_B \times \langle k'' |_A \langle n' |_B (X_{AB}) | k' \rangle_A | n \rangle_B = \\ &\sum_n \sum_{n'} \langle n |_B Y_B | n' \rangle_B \times \langle k |_A \langle n' |_B (X_{AB}) | k' \rangle_A | n \rangle_B. \end{aligned}$$

$$\langle k |_A rhs | k' \rangle_A = \sum_n \langle k |_A \langle n |_B (X_{AB} Y_B) | k' \rangle_A | n \rangle_B =$$

$$\sum_n \sum_{k''} \sum_{n'} \langle k |_A \langle n |_B X_{AB} | k'' \rangle_A | n' \rangle_B \times \langle k'' |_A \langle n' |_B (I_A \otimes Y_B) | k' \rangle_A | n \rangle_B =$$

$$\sum_n \sum_{n'} \langle k |_A \langle n |_B X_{AB} | k' \rangle_A | n' \rangle_B \times \langle n' |_B Y_B | n \rangle_B.$$

Finally, we exchange the order of the two factors and the two mute indices n and n' to obtain

$$\langle k |_A rhs | k' \rangle_A = \sum_{n'} \sum_n \langle n |_B Y_B | n' \rangle_B \times \langle k |_A \langle n' |_B (X_{AB}) | k' \rangle_A | n \rangle_B.$$

Thus, we see that $lhs = rhs$ as claimed.

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